Contemporary Mathematics
Volume 35, 1984

APPROXIMATING CELL-LIKE MAPS OF S^4 BY HOMEOMORPHISMS

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ABSTRACT. We present a proof of FREEDMAN'S APPROXIMATION THEOREM: A surjective map $f:S^n \to S^n$ can be approximated by homeomorphisms if (1) $S(f) = \{y \in S^n : \operatorname{diam} f^{-1}(y) > 0\}$ is a nowhere dense subset of S^n, and (2) $\{f^{-1}(y) : y \in S(f)\}$ is a null collection (for every $\varepsilon > 0$, $\{y \in S(f) : \operatorname{diam} f^{-1}(y) \geq \varepsilon\}$ is a finite set). We then show that these hypotheses can be weakened as follows. A suggestion of R. D. Edwards allows us to replace (1) by: f has a bald spot (there is a non-empty open subset U of S^n such that $f|f^{-1}U : f^{-1}U \to U$ is a homeomorphism). (2) can be replaced by: $S(f)$ is a tame zero-dimensional subset of S^n (each point of $S(f)$ has arbitrarily small collared n-cell neighborhoods whose boundaries miss $S(f)$).

1. INTRODUCTION

Let X and Y be compact metric spaces, and let $f:X \to Y$ be a map. For $\varepsilon > 0$, a map $g:X \to Y$ is within ε of f if $d(f(x),g(x)) < \varepsilon$ for every $x \in X$. f can be approximated by homeomorphisms if for every $\varepsilon > 0$, there is a homeomorphism from X to Y which is within ε of f. If the space Y is locally contractible (for instance, if Y is a manifold), then an easily verified necessary condition for f to be approximable by homeomorphisms is that for each $y \in Y$, $f^{-1}(y)$ contracts to a point in each of its neighborhoods in X. This leads us to the following definition. A subset of X is cell-like if it contracts to a point in each of its neighborhoods in X. The Whitehead continuum is a cell-like (but not contractible) subset of S^3 of great renown. The map $f:X \to Y$ is cell-like if $f^{-1}(y)$ is a cell-like subset of X for each $y \in Y$. We shall consider the question of whether a given cell-like map between spheres can be approximated by homeomorphisms.

For $n \neq 4$, the approximation theorems of [A] and [S] imply that any cell-like map $f:S^n \to S^n$ can be approximated by homeomorphisms. The proofs of these results depend on techniques which are specific to dimension 3 or to high dimensions, and which until recently had no analogues in dimension 4. M. Freedman's August, 1981 construction of topological 2-handles in 4-manifilds [F] changed this situation dramatically. Indeed, in July, 1982 (during the conference whose Proceedings these are), F. Quinn used Freedman's work to obtain a general theorem [Q] which has as a corollary that any cell-like map between

0271-4132/84 $1.00 + $.25 per page

4-spheres can be approximated by homeomorphisms. Freedman's construction depends crucially on the fact that certain special types of cell-like maps between spheres can be approximated by homeomorphisms. We call this fact Freedman's Approximation Theorem. Its ingenious proof (which works in all dimensions) is expounded below, along with proofs of several extensions. Thus the general result that any cell-like map between 4-spheres can be approximated by homeomorphisms follows from Quinn's work which in turn depends on the special case established by Freedman's Approximation Theorem.

To state Freedman's Approximation Theorem and its extensions, we require the following definitions. Again let X and Y be compact metric spaces, and let $f: X \to Y$ be a map. The singular set of f, denoted $S(f)$, is the set $\{y \in Y : f^{-1}(y)$ contains more than one point$\}$. Observe that for every $\varepsilon > 0$, the set $\{y \in Y : \operatorname{diam} f^{-1}(y) \geq \varepsilon\}$ is compact. Since $S(f) = \bigcup_{i=1}^{\infty} \{y \in Y : \operatorname{diam} f^{-1}(y) \geq 1/i\}$, we conclude that $S(f)$ is σ-compact. A subset of Y is nowhere dense if its closure has empty interior. A collection $\mathscr{C}$ of subsets of X is a null collection if for every $\varepsilon > 0$, $\{C \in \mathscr{C} : \operatorname{diam} C \geq \varepsilon\}$ is a finite set. Thus, if $\{f^{-1}(y) : y \in S(f)\}$ is a null collection of subsets of X, then $S(f)$ is a countable set. f has a bald spot if there is a non-empty open subset U of Y such that $f|f^{-1}U : f^{-1}U \to U$ is a homeomorphism. Thus, f has a bald spot if it is surjective and if $c\ell\,(f) \neq Y$.

Let M be an n-manifold. An n-cell C in $\operatorname{int} M$ is collared if there is an embedding of $\partial C \times [0,1]$ in $M - \operatorname{int} C$ which takes $\partial C \times \{0\}$ onto ∂C. A σ-compact subset S of $\operatorname{int} M$ is tame zero-dimensional in M if each point of S has arbitrarily small collared n-cell neighborhoods whose boundaries miss S (in other words, for every $y \in S$ and every neighborhood U of y in M, there is a collared n-cell C in M such that $y \in \operatorname{int} C$, $C \subset U$ and $(\partial C) \cap S = \emptyset$).

We shall present proofs of the following theorems.

THEOREM 1: FREEDMAN'S APPROXIMATION THEOREM. *A surjective map $f: S^n \to S^n$ can be approximated by homeomorphisms if* (1) $S(f)$ *is a nowhere dense subset of S^n and* (2) $\{f^{-1}(y) : y \in S(f)\}$ *is a null collection.*

A suggestion of R. D. Edwards for reorganizing Freedman's proof of Theorem 1 leads to a proof of:

THEOREM 2. *A map $f: S^n \to S^n$ can be approximated by homeomorphisms if* (1) f *has a bald spot and* (2) $S(f)$ *is a countable subset of S^n.*

Finally an "amalgamation procedure" combines with a shrinking principle due to R. H. Bing to yield:

THEOREM 3. *A map $f: S^n \to S^n$ can be approximated by homeomorphisms if* (1) f *has a bald spot and $S(f)$ is a tame zero-dimensional subset of S^n.*

Before embarking on the proofs of these theorems, we make several remarks. First, we note that the surjectivity hypothesis in Theorem 1 would be redundant in Theorems 2 and 3, because the bald spot hypothesis implies that f is degree 1 and, thus, surjective.

Second, we note that although the hypotheses of these three theorems do not explicitly state that f is a cell-like map, they easily imply that it is. For let $y \in S(f)$. The tame zero-dimensionality of $S(f)$ implies that $f^{-1}(y)$ has arbitrarily tight closed neighborhoods whose frontiers are $(n-1)$-spheres. These neighborhoods must be contractible. Hence $f^{-1}(y)$ is cell-like.

In his construction of topological 2-handles in 4-manifolds, Freedman applies Theorem 1 at a crucial point to a map $f: S^4 \to S^4$. The validity of this application depends on $S(f)$ being nowhere dense in S^4. In Freedman's context, $S(f)$ is nowhere dense because its closure is a 1-dimensional subset of S^4.

We close this section with some comments about the proof of Theorem 1, including a comparison to M. Brown's proof of the Generalized Schoenflies Theorem.

Freedman's Approximation Theorem might be regarded as a generalization of [Br], because Brown's method of proof implicitly establishes the following:

THEOREM 0. <u>A surjective map $f: S^n \to S^n$ can be approximated by homeomorphisms if $S(f)$ is a finite set.</u>

There is a superficial resemblance between the techniques used by Brown to prove Theorem 0 and those used by Freedman for Theorem 1. We find it instructive to review the outline of Brown's argument for Theorem 0, to contrast the two methods of proof, and to focus on the difficulties that must be overcome by any proof of Theorem 1 which don't arise in the proof of Theorem 0.

To review Brown's proof of Theorem 0, consider a surjective map $f: S^n \to S^n$ with a finite singular set. First, one argues by induction on the number of points in $S(f)$ that for each $y \in S(f)$, $f^{-1}(y)$ is a cellular subset of S^n ($f^{-1}(y)$ has arbitrarily tight n-cell neighborhoods in S^n). (This is a slight oversimplification; in the actual proof, one must work with a map $f: B^n \to S^n$ such that $S(f)$ is finite and disjoint from $f(\partial B^n)$.) Second, one uses the cellularity of the preimages of the points of $S(f)$ to "shrink" these sets independently to produce a homeomorphism approximating f. Neither of these steps is possible under the hypotheses of Freedman's Approximation Theorem. First, since $S(f)$ may be countably infinite, no induction argument will establish the cellularity of the preimages of the points of $S(f)$. Second, even if the cellularity of the preimages of the points in $S(f)$ is given in advance, they cannot be shrunk independently. The problem is that a motion which shrinks the larger preimage sets small may necessarily stretch some of the smaller sets. The classic example of this phenomenon is Bing's null cellular decomposition of S^3 [B2] whose quotient map is not approximable by homeomorphisms because its quotient space is not S^3.

In Freedman's proof of Theorem 1, the cellularity of the preimages of the points of $S(f)$ is never established in the course of the argument. It follows only after the proof is finished as a consequence of the conclusion of the theorem.

Freedman's proof is not a traditional "shrinking argument" in the sense of decomposition space theory. It has a more complex logical structure. Instead of shrinking the large point inverses of f, it uses a replication device which makes the large point images of f disappear at the cost of complicating the logical framework of the argument. Specifically, the replication device forces the use of relations which are neither maps nor their inverses. In fact, the approximating homeomorphism which is the goal of the proof arises as the limit of such relations. For this reason, simple techniques for manipulating relations appear.

2. TWO LEMMAS

We introduce some terminology and establish two lemmas which find use in the proofs of Theorems 1 and 2.

The first lemma is a general position property of countable subsets of manifolds. The following remarks about the homeomorphism group of a compactum are included to simplify its proof.

Suppose X is a compact space with metric ρ. Let $\mathcal{H}(X)$ denote the space of homeomorphisms of X with the compact-open topology. (One basis for the compact-open topology on $\mathcal{H}(X)$ consists of all sets of the form $\{h \in \mathcal{H}(X) : h \subset 0\}$ where 0 varies over the open subsets of $X \times X$.) The compact-open topology on $\mathcal{H}(X)$ is induced by the "supremum metric" σ which is defined by $\sigma(g,h) = \sup\{\rho(g(x),h(x)) : x \in X\}$. Although σ is generally not a complete metric on $\mathcal{H}(X)$, a complete metric τ on $\mathcal{H}(X)$ is easily produced in terms of σ by the formula $\tau(g,h) = \sigma(g,h) + \sigma(g^{-1},h^{-1})$. For a subset A of X, define $\mathcal{H}(X,A) = \{h \in \mathcal{H}(X) : h|A = 1|A\}$. If $A \subset X$, then $\mathcal{H}(X,A)$ is a closed subset of $\mathcal{H}(X)$; hence, the complete metric τ on $\mathcal{H}(X)$ restricts to a complete metric on $\mathcal{H}(X,A)$.

Two subsets S and T of a metric space X are <u>separated</u> in X if $(c\ell S) \cap T = \emptyset = S \cap (c\ell T)$ (or equivalently if there are disjoint open subsets U and V of X such that $S \subset U$ and $T \subset V$).

LEMMA 1. *Let M be a compact manifold.*

(1) *If S is a countable subset of $\operatorname{int} M$ and T is the union of a countable number of nowhere dense subsets of M, then $1|M$ can be approximated by homeomorphisms h of M such that $h(S) \cap T = \emptyset$ and $h|\partial M = 1|\partial M$.*

(2) *If S and T are countable nowhere dense subsets of $\operatorname{int} M$, then $1|M$ can be approximated by homeomorphisms h of M such that $h(S)$ and T are*

separated in M and $h|\partial M = 1|\partial M$.

PROOF OF (1). Let $S = \{s_i\}$, and let $T = \bigcup_{j=1}^{\infty} T_j$ where each T_j is a nowhere dense subset of M. For each $i \geq 1$, let $U_{i,j} = \{h \in \mathscr{H}(M,\partial M): h(s_i) \notin c\ell T_j\}$. It is easily seen that each $U_{i,j}$ is a dense open subset of $\mathscr{H}(H,\partial M)$. Since $\mathscr{H}(M,\partial M)$ has a complete metric, we conclude via the Baire Category Theorem that $\bigcap_{i=1}^{\infty} {}_{j=1}^{\infty} U_{i,j}$ is a dense subset of $\mathscr{H}(M,\partial M)$. Statement (1) now follows because $1|M$ can be approximated by elements of $\bigcap_{i=1}^{\infty}{}_{j=1}^{\infty} U_{i,j}$.

PROOF OF (2). Assume $S = \{s_i\}$ and $T = \{t_j\}$ are countable nowhere dense subsets of $\operatorname{int} M$. For each $i \geq 1$, let $U_i = \{h \in \mathscr{H}(M,\partial M): h(s_i) \notin c\ell T\}$ and let $V_i = \{h \in \mathscr{H}(M,\partial M): t_i \notin h(c\ell S)\}$. It is easily seen that each U_i and each V_i are dense open subsets of $\mathscr{H}(M,\partial M)$. As above, since $\mathscr{H}(M,\partial M)$ has a complete metric, the Baire Category Theorems implies that $\bigcap_{i=1}^{\infty}(U_i \cap V_i)$ is a dense subset of $\mathscr{H}(M,\partial M)$. Statement (2) now follows because $1|M$ can be approximated by elements of $\bigcap_{i=1}^{\infty}(U_i \cap V_i)$. ⦀

The second lemma concerns relations. It is used in the proofs of Theorems 1 and 2 to guarantee that the sequences of relations which are produced in these proofs converge to homeomorphisms. In order to streamline the next lemma and the proofs of Theorems 1 and 2, we now establish some notation for relations which generalizes the usual functional notation.

Let $R \subset X \times Y$; i.e., R is a relation from the set X to the set Y. Define

$$R^{-1} = \{(y,x) \in Y \times X: (x,y) \in R\}.$$

If $S \subset Y \times Z$, define

$$S \circ R = \{(x,z) \in X \times Z: (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in Y\}.$$

If $x \in X$, define $R(x) = \{y \in Y: (x,y) \in R\}$. Thus for $y \in Y$, $R^{-1}(y) = \{x \in X: (x,y) \in R\}$. If $x \in X$, then $R(x)$ is called a <u>point image</u> of R; and if $y \in Y$, then $R^{-1}(y)$ is called a <u>point inverse</u> of R. If $A \subset X$, define $R(A) = \bigcup\{R(x): x \in A\}$ and define $R|A = R \cap (A \times Y)$.

LEMMA 2. *Let R be a closed subset of $X \times Y$ where X and Y are compact metric spaces. Suppose T is a closed subset of X, $\epsilon > 0$ and $\operatorname{diam} R(x) < \epsilon$ for every $x \in X-T$. Then there is a closed subset N of $X \times Y$ such that $R|X-T \subset \operatorname{int} N$, $\operatorname{diam} N(x) < \epsilon$ for every $x \in X-T$, and $N|T = R|T$.*

PROOF. Let $M_1 \supset M_2 \supset M_3 \supset \cdots$ be a decreasing sequence of closed neighborhoods of R in $X \times Y$ such that $\bigcap_{i=1}^{\infty} M_i = R$. We assert that if A is a compact subset of $X-T$, then for some $i \geq 1$, $\operatorname{diam} M_i(x) < \epsilon$ for every $x \in A$. For otherwise, there are sequences $\{(x_i,y_i)\}$ and $\{(x_i,z_i)\}$ in $A \times Y$ such that for each $i \geq 1$, (x_i,y_i) and (x_i,z_i) lie in M_i and $\operatorname{diam}\{y_i,z_i\} \geq \epsilon$. Since A and Y are compact, then by passing to subsequences, we can assume that the sequence $\{x_i\}$ converges to a point x in A, and that the sequences $\{y_i\}$

and $\{z_i\}$ converge to points y and z, respectively, in Y. Consequently, $\operatorname{diam}\{y,z\} \geq \varepsilon$. Also since $R = \cap_{i=1}^{\infty} M_i$, it follows that (x,y) and (x,z) belong to R. Hence y and z belong to $R(x)$. Since $\operatorname{diam} R(x) < \varepsilon$, we have a contradiction. Our assertion follows.

Let $\{A_i\}$ be a sequence of compact subsets of $X-T$ such that $A_i \subset \operatorname{int} A_{i+1}$ for each $i \geq 1$, and $\cup_{i=1}^{\infty} A_i = X-T$. Set $A_0 = \emptyset$. The above assertion implies that by passing to an appropriate subsequence of $\{M_i\}$, we obtain a decreasing sequence $N_1 \supset N_2 \supset N_3 \supset \cdots$ of closed neighborhoods of R such that $\cap_{i=1}^{\infty} N_i = R$ and for each $i \geq 1$, $\operatorname{diam} N_i(x) < \varepsilon$ for every $x \in A_i$. Set $N = (\cup_{i=1} N_i | A_i) \cup (R|T)$. We find it convenient to define, for each $i \geq 1$, a closed neighborhood P_i of R in $X \times Y$ by setting $P_i = (\cup_{j=1}^{i-1} N_j | A_j) \cup N_i$. Then $N = \cap_{i=1}^{\infty} P_i$; so N is a closed subset of $X \times Y$. For each $i \geq 1$, since $P_i | A_i = N | A_i$, then $R|\operatorname{int} A_i \subset \operatorname{int} P_i | \operatorname{int} A_i \subset \operatorname{int} N$; it follows that $R|X-T \subset \operatorname{int} N$. For each $i \geq 1$, if $x \in A_i - A_{i-1}$, then $\operatorname{diam} N(x) = \operatorname{diam} N_i(x) < \varepsilon$; hence $\operatorname{diam} N(x) < \varepsilon$ for every $x \in X-T$. Clearly $N|T = R|T$. ∎

3. FREEDMAN'S APPROXIMATION THEOREM

A map $f\colon B^n \to B^n$ is <u>admissible</u> if $f|\partial B^n = 1|\partial B^n$, $S(f)$ is a nowhere dense subset of B^n, $c\ell S(f) \subset \operatorname{int} B^n$, and $\{f^{-1}(y)\colon y \in S(f)\}$ is a null collection.

We shall now argue that Freedman's Approximation Theorem reduces to:

THEOREM 1A. *Every admissible map $f\colon B^n \to B^n$ can be approximated by homeomorphisms.*

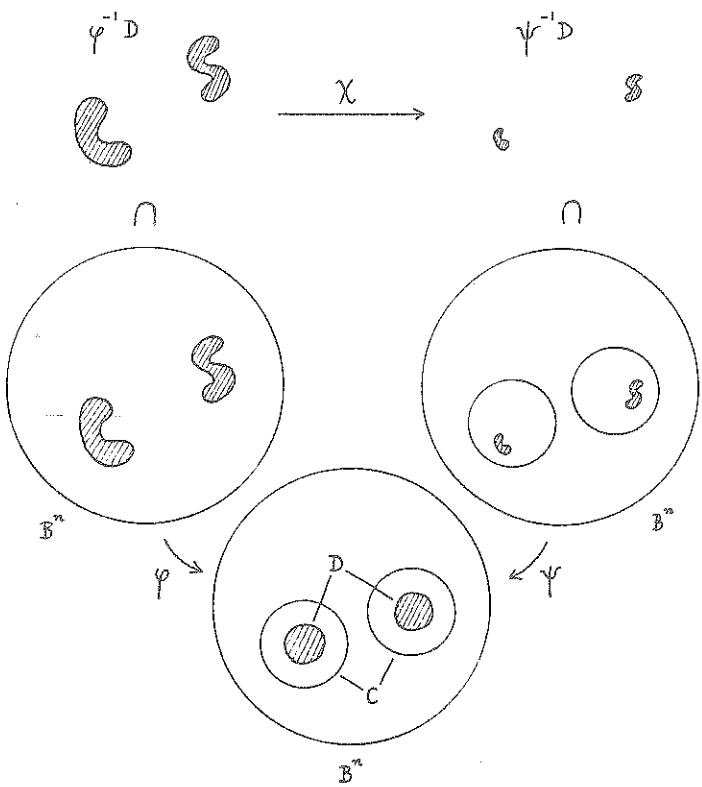
PROOF OF FREEDMAN'S APPROXIMATION THEOREM FROM THEOREM 1A. Assume Theorem 1A. Suppose $f\colon S^n \to S^n$ is a map with a nowhere dense singular set whose point inverses form a null collection. Let $\varepsilon > 0$. Since $S(f)$ is nowhere dense, there is a collared n-cell C in $S^n - c\ell S(f)$ of diameter $< \varepsilon$. The Generalized Schoenflies Theorem [Br] produces homeomorphisms $\varphi\colon B^n \to c\ell(S^n - f^{-1}C)$ and $\psi\colon B^n \to c\ell(S^n - C)$; furthermore ψ can be adjusted so that $\psi|\partial B^n = f \circ \varphi|\partial B^n$. Then $\psi^{-1} \circ f \circ \varphi\colon B^n \to B^n$ is an admissible map. The uniform continuity of ψ provides a $\delta > 0$ so that ψ carries any set of diameter $< \delta$ to a set of diameter $< \varepsilon$. Theorem 1A gives us a homeomorphism $g\colon B^n \to B^n$ which is within δ of $\psi^{-1} \circ f \circ \varphi$. It follows that $\psi \circ g \circ \varphi^{-1}\colon c\ell(S^n - f^{-1}C) \to c\ell(S^n - C)$ is a homeomorphism which is within ε of $f|c\ell(S^n - f^{-1}C)$. Since $\psi \circ g \circ \varphi^{-1}$ maps $f^{-1}(\partial C)$ homeomorphically onto ∂C, and since $\operatorname{diam} C < \varepsilon$, then $\psi \circ g \circ \varphi^{-1}$ extends to a homeomorphism of S^n which is within ε of f.

The geometric idea lying at the heart of the proof of Freedman's Approximation Theorem is a very simple replication device which is crystallized in the following lemma. In this lemma, the pre-image pattern of the given admissible map φ on $\varphi^{-1}D$ is replicated by a new admissible map ψ on $\psi^{-1}D$; and the replication is witnessed by a homemorphism $\chi\colon \varphi^{-1}D \to \psi^{-1}D$ such that

$\psi \circ \chi = \varphi|\varphi^{-1}D$. We foreshadow the proof of the theorem to the extent of remarking that this replication allows us to replace the map φ by a relation R which equals χ on $\varphi^{-1}D$ and which equals $\psi^{-1} \circ \varphi$ on $B^n - \varphi^{-1}D$. R represents an improvement over φ in that it has no non-trivial point inverses in $\varphi^{-1}D$. The apparent disadvantage of this procedure is that it exchanges a map for a relation.

We need the following terminology for the lemma. Let $|\ |$ denote the Euclidean norm on $\mathbb{R}^n$; i.e., $|x| = (\Sigma_{i=1}^n x_i^2)^{\frac{1}{2}}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. An n-cell C in $\mathbb{R}^n$ is round if there is a point x in $\mathbb{R}^n$ called the center of C and a positive number r called the radius of C such that $C = \{y \in \mathbb{R}^n : |x-y| \leq r\}$. Note that if C is a round n-cell in B^n and D is a compactum in $\operatorname{int} C$, then a homeomorphism $\sigma: C \to B^n$ such that $\sigma|D = 1|D$ is easily obtained by sliding along the radial structure emanating from the center of C.

LEMMA 3 (THE REPLICATION DEVICE). Suppose $\varphi: B^n \to B^n$ is an admissible map, C and D are each the union of a finite number of disjoint round n-cells in $\operatorname{int} B^n$, $D \subset \operatorname{int} C$ and $S(\varphi) \cap \partial D = \emptyset$. Then there is an admissible map $\psi: B^n \to B^n$ and a homeomorphism $\chi: \varphi^{-1}D \to \psi^{-1}D$ such that $\psi \circ \chi = \varphi|\varphi^{-1}D$, $\psi(\operatorname{int} C) = \operatorname{int} C$, ψ restricts to the identity on $B^n - \operatorname{int} C$, $S(\psi) \cap \partial D = \emptyset$, and $S(\varphi) - D$ and $S(\psi) - D$ are separated.



PROOF. $C = \bigcup_{i=1}^{k} C_i$ where each C_i is a round n-cell in $\text{int}\, B^n$. We shall define ψ so that for each i, $1 \leq i \leq k$, $\psi | C\colon C_i \to C_i$ is a minaturized replica of φ.

Let $1 \leq i \leq k$. Let $D_i = D \cap C_i$. We shall construct a homeomorphism $\tau_i\colon C_i \to B^n$ such that $\tau_i | D_i = 1 | D_i$, and $S(\varphi) - D_i$ and $\tau_i^{-1}(S(\varphi)) - D_i$ are separated. To begin, there is a homeomorphism $\sigma_i\colon C_i \to B^n$ such that $\sigma_i | D_i = 1 | D_i$. Since $S(\varphi)$ and $\sigma_i^{-1}(S(\varphi))$ are countable and nowhere dense, then we can apply Lemma 1 in $C_i - \text{int}\, D_i$ to obtain a homeomorphism λ_i of C_i which restricts to the identity on $D_i \cup (\partial C_i)$ such that $\lambda_i(\sigma_i^{-1}(S(\varphi)) - D_i$ and $S(\varphi) \cap (\text{int}\, C_i - D_i)$ are separated. Since $c\ell S(\varphi) \subset \text{int}\, B^n$, then $c\ell(\lambda_i \circ \sigma_i^{-1}(S(\varphi)) \subset \text{int}\ C_i$. It follows that $\lambda_i \circ \sigma_i^{-1}(S(\varphi)) - D_i$ and $S(\varphi) - D_i$ are separated. The desired homeomorphism τ_i is obtained by setting $\tau_i = \sigma_i \circ \lambda_i^{-1}$.

Define the map $\psi\colon B^n \to B^n$ by

$$\psi = \begin{cases} \tau_i^{-1} \circ \varphi \circ \tau_i & \text{on } C_i \quad \text{for } 1 \leq i \leq k \\ 1 \quad \text{on} \quad B^n - \text{int}\, C & \end{cases}$$

Since $S(\psi) = \bigcup_{i=1}^{k} \tau_i^{-1}(S(\varphi))$, it is easily verified that ψ is an admissible map, $S(\psi) \cap \partial D = \emptyset$, and $S(\varphi) - D$ and $S(\psi) - D$ are separated.

Since $\psi^{-1} D_i = \tau_i^{-1}(\varphi^{-1} D_i)$ for $1 \leq i \leq k$, then a homeomorphism $\chi\colon \varphi^{-1} D \to \psi^{-1} D$ is defined by setting $\chi | \varphi^{-1} D_i = \tau_i^{-1} | \varphi^{-1} D_i$ for $1 \leq i \varepsilon k$. Clearly $\psi \circ \chi = \varphi | \varphi^{-1} D$. ▥

PROOF OF THEOREM 1A. The proof is inductive. The induction step, which has a rather technical statement, is isolated in Lemma 4 below.

We begin by describing the strategy of the proof. Let $f\colon B^n \to B^n$ be an admissible map. Let $\varepsilon > 0$. Set $N_0 = \{(x,y) \in B^n \times B^n\colon |f(x) - y| \leq \varepsilon\}$. N_0 is a closed neighborhood of f in $B^n \times B^n$. Our goal is to produce a homeomorphism $h\colon B^n \to B^n$ such that $h \subset N_0$. This will be accomplished by constructing a decreasing sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of closed subsets of $B^n \times B^n$ with the property that for each $i \geq 1$ and every $x \in B^n$, $N_i(x)$ and $N_i^{-1}(x)$ are non-empty sets of diameter $< 1/i$. Upon setting $h = \bigcap_{i=0}^{\infty} N_i$, we see that $h\colon B^n \to B^n$ is a bijection which is, in fact, a homeomorphism because it is a closed subset of $B^n \times B^n$.

Before we give more details, we find it convenient to introduce one more bit of terminology. A relation $R \subset B^n \times B^n$ is <u>admissible</u> if

$$R = h \cup g^{-1} \circ f | f^{-1}(B^n - \text{int}\, A)$$

where

(1) $f: B^n \to B^n$ and $g: B^n \to B^n$ are admissible maps,

(2) A is the union of a finite number of disjoint round n-cells in $\operatorname{int} B^n$ such that $(S(f) \cup S(g)) \cap \partial A = \emptyset$ and $S(f) - A$ and $S(g) - A$ are separated, and

(3) $h: f^{-1}A \to g^{-1}A$ is a homeomorphism such that $g \circ h = f|f^{-1}A$.

$$\begin{array}{ccc} f^{-1}A & \xrightarrow{\ h\ } & g^{-1}A \\ \cap & & \cap \\ B^n & \xrightarrow{\ R\ } & B^n \\ & {}_{f}\searrow \quad \swarrow_{g} & \\ & B^n & \end{array}$$

Let $R = h \cup g^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A)$ be an admissible relation in $B^n \times B^n$, where f, g, h and A are as prescribed above. We observe that R is a closed subset of $B^n \times B^n$. This is a consequence of two statements. First f, g and h are compact because each is a continuous function with compact domain and range. Second, the operations of inversion, composition and restriction over a closed set all transform compact relations into compact relations. We also observe that the inverse of an admissible relation is admissible.

We now give the details of the proof of Theorem 4. Set $R_0 = f$; then R_0 is an admissible relation (with $g = 1|B^n$, $A = \emptyset$ and $h = \emptyset$). The closed neighborhood N_0 of R_0 has already been defined. We shall construct a sequence $\{R_i\}$ of admissible relations in $B^n \times B^n$ and a sequence $\{N_i\}$ of closed subsets of $B^n \times B^n$ such that for each $i \geq 1$ the following conditions hold.

(1_i) $R_i \subset \operatorname{int} N_{i-1}$, $\operatorname{diam} R_i^{-1}(y) < 1/i+1$ for every $y \in B^n$ when i is odd, and $\operatorname{diam} R_i(x) < 1/i+1$ for every $x \in B^n$ when i is even.

(2_i) N_i is a closed neighborhood of R_i in $B^n \times B^n$ such that $N_i \subset N_{i-1}$, $\operatorname{diam} N_i^{-1}(y) < 1/i+1$ for every $y \in B^n$ when i is odd, and $\operatorname{diam} N_i(x) < 1/i+1$ for every $x \in B^n$ when i is even.

R_0 and N_0 are already in hand. We proceed inductively. Let $i \geq 1$ and assume we have an admissible relation R_{i-1} and a closed neighborhood N_{i-1} of R_{i-1} in $B^n \times B^n$. We obtain R_i satisfying (1_i) via Lemma 4 below. When i is <u>odd</u>: we apply Lemma 4 by substituting $(R_{i-1}, 1/i+1, N_{i-1})$ for (R, ε, N); then Lemma 4 produces R_*, and we set $R_i = R_*$. When i is <u>even</u>: we apply Lemma 4 by substituting $(R_{i-1}^{-1}, 1/i+1, N_{i-1}^{-1})$ for (R, ε, N); then Lemma 4 produces R_*, and we set $R_i = R_*^{-1}$.

Next we use Lemma 2 to obtain N_i satisfying (2_i). When i is <u>odd</u>: we apply Lemma 2 by substituting $(B^n, B^n, R_i^{-1}, \emptyset, 1/i+1)$ for $(X, Y, R, T, \varepsilon)$; then Lemma 2 produces N, and we set $N_i = N^{-1} \cap N_{i-1}$. When i is <u>even</u>: we apply Lemma 2 by substituting $(B^n, B^n, R_i, \emptyset, 1/i+1)$ for $(X, Y, R, T, \varepsilon)$; then Lemma 2 produces N, and we set $N_i = N \cap N_{i-1}$.

Let $i \geq 2$. Since R_i is admissible, then $R_i(x)$ and $R_i^{-1}(x)$ are non-empty for every $x \in B^n$. Since $R_i \subset N_i \subset N_{i-1}$, then (2_{i-1}) and (2_i) imply that $N_i(x)$ and $N_i^{-1}(x)$ are non-empty sets of diameter $< 1/i$ for every $x \in B^n$. |||

LEMMA 4. <u>If</u> $R \subset B^n \times B^n$ <u>is an admissible relation</u>, $\varepsilon > 0$, <u>and</u> N <u>is a closed neighborhood of</u> R <u>in</u> $B^n \times B^n$, <u>then there is an admissible relation</u> $R_* \subset B^n \times B^n$ <u>such that</u> $\operatorname{diam} R_*^{-1}(y) < \varepsilon$ <u>for every</u> $y \in B^n$ <u>and</u> $R_* \subset \operatorname{int} N$.

PROOF. Since R is admissible, then $R = h \cup g^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A)$, where f, g, A and h are as prescribed in the definition of "admissible relation". Let $Z = \{z \in S(f): \operatorname{diam} f^{-1}(z) \geq \varepsilon\} - A$. Z is a finite subset of $\operatorname{int} B^n$ because f is an admissible map. The significance of Z is that $\{f^{-1}(z): z \in Z\} = \{R^{-1}(y): y \in B^n$ and $\operatorname{diam} R^{-1}(y) \geq \varepsilon\}$, and the latter set is precisely the set of point inverses of R whose diameters must be reduced.

Here is a rough idea of how we proceed. We enclose Z in the union D of a finite number of small disjoint round n-cells in $\operatorname{int} B^n$. Then we use the Replication Device (Lemma 3) to modify the map g so that the preimage pattern of f on $f^{-1}D$ is replicated by g on $g^{-1}D$. This allows us to redefine R on $f^{-1}D$ so that it carries $f^{-1}D$ <u>homeomorphically</u> onto $g^{-1}D$. In this way, the large point inverses of R simply vanish at the expense of complicating the structure of the map g.

There is a finite collection $C_1, C_2, \ldots, C_k$ of disjoint round n-cells in $\operatorname{int} B^n$ such that if $C = \bigcup_{i=1}^k C_i$, then $Z \subset \operatorname{int} C$, $C \cap (A \cup c\ell(S(g))) = \emptyset$, and $f^{-1}C_i \times g^{-1}C_i \subset \operatorname{int} N$ for $1 \leq i \leq k$. The second condition can be achieved because $S(f) - A$ and $S(g) - A$ are separated, and Z is a finite subset of $S(f) - A$. The third condition holds automatically for C_i's of sufficiently small diameter because for each $z \in Z$, $f^{-1}(z) \times g^{-1}(z) = R|f^{-1}(z) \subset \operatorname{int} N$. (The third condition will be used to insure that $R_* \subset \operatorname{int} N$.) Since $S(f)$ is a countable set, then for each i, $1 \leq i \leq k$, there is a round n-cell D_i such that $D_i \subset \operatorname{int} C_i$ and if $D = \bigcup_{i=1}^k D_i$, then $Z \subset \operatorname{int} D$ and $S(f) \cap \partial D = \emptyset$.

We now apply Lemma 3 with f in the role of φ, to obtain an admissible map $\psi: B^n \to B^n$ and a homeomorphism $\chi: f^{-1}D \to \psi^{-1}D$ such that $\psi \circ \chi = f|f^{-1}D$, $\psi(\operatorname{int} C) = \operatorname{int} C$, $\psi = 1$ on $B^n - \operatorname{int} C$, $S(\psi) \cap \partial D = \emptyset$, and $S(f) - D$ and $S(\psi) - D$ are separated.

We define the map $g_*: B^n \to B^n$ by $g_* = \psi \circ g$. Since $S(\psi) \subset C$ and $C \cap c\ell S(g) = \emptyset$, then evidently $S(g_*) = S(\psi) \cup S(g)$ and g_* is an admissible map.

We set $A_* = A \cup D$. Then A_* is the union of a finite number of disjoint round n-cells in $\operatorname{int} B^n$. It is easily verified that $(S(f) \cup S(g_*)) \cap \partial A_* = \emptyset$ and that $S(f) - A_*$ and $S(g_*) - A_*$ are separated.

Since $C \cap (A \cup cl(S(g))) = \emptyset$ and $\psi^{-1}D \subset C$, then $g_*^{-1}A_* = g^{-1}A \cup g^{-1}(\psi^{-1}D)$ and $g^{-1}|\psi^{-1}D$ is a homeomorphism. Hence a homeomorphism $h_*: f^{-1}A_* \to g_*^{-1}A_*$ is defined by setting $h_*|f^{-1}A = h$ and $h_*|f^{-1}D = g^{-1} \circ \chi$. It follows easily that $g_* \circ h_* = f|f^{-1}A_*$.

Finally, an admissible relation $R_* \subset B^n \times B^n$ is defined by setting $R_* = h_* \cup g_*^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A_*)$.

Note that $R_*^{-1} = h_*^{-1} \cup f^{-1} \circ g_*|g_*^{-1}(B^n - \operatorname{int} A_*)$. Hence, if $y \in B^n$ and $\operatorname{diam} R_*^{-1}(y) > 0$ then $y \in g_*^{-1}(B^n - \operatorname{int} A_*)$ and $R_*^{-1}(y) = f^{-1}(g_*(y))$. Z, D and A_* are chosen to guarantee that $\{z \in S(f): \operatorname{diam} f^{-1}(z) \geq \varepsilon\} \subset \operatorname{int} A_*$. Since $g_*(y) \notin \operatorname{int} A_*$, it follows that $\operatorname{diam} f^{-1}(g_*(y)) < \varepsilon$. Thus $\operatorname{diam} R_*^{-1}(y) < \varepsilon$.

Lastly, we demonstrate that $R_* \subset \operatorname{int} N$. First, since $g_*^{-1} = g^{-1}$ on $B^n - \operatorname{int} C$ and $h_* = h$ on $f^{-1}A$, it follows that $R_*|f^{-1}(B^n - \operatorname{int} C) = R|f^{-1}(B^n - \operatorname{int} C) \subset \operatorname{int} N$. Second, we use the equation $g_* \circ h_* = f|f^{-1}A_*$ to deduce that $h_* \subset g_*^{-1} \circ f$; therefore, $R_* \subset g_*^{-1} \circ f$. For $1 \leq i \leq k$, since $\psi(C_i) = C_i$, then $g_*^{-1}(C_i) = g^{-1}(C_i)$. Therefore, for $1 \leq i \leq k$,

$$R_*|f^{-1}C_i \subset g_*^{-1} \circ f|f^{-1}C_i \subset f^{-1}(C_i) \times g_*^{-1}(C_i) = f^{-1}(C_i) \times g^{-1}(C_i) \subset \operatorname{int} N.$$

Consequently, $R_*|f^{-1}C \subset \operatorname{int} N$. It is now evident that $R_* \subset \operatorname{int} N$. ▯

4. MAPS WITH A BALD SPOT

The proofs of Theorems 1 and 2 are quite similar, and we rely on the reader's familiarity with the proof of Theorem 1 at several points in the proof of Theorem 2. We feel the reader may be aided, if we pause here to draw some comparisons between the two proofs.

The proof of Theorem 1 produces a homeomorphism by an infinite process which alternates between excising point images and point inverses of an admissible relation. Successive steps in this process apply the replication device to "opposite sides" of the relation. The ability to "switch sides" repeatedly depends on the point images and point inverses of the relation being separated (when viewed in the appropriate space). Disjointness alone is not sufficient. This separation can be achieved only because the singular set of the original map is nowhere dense.

When the singular set of the original map is countable but not necessarily nowhere dense (as in Theorem 2), then the replication device yields relations whose point images and point inverses can be made disjoint but can't necessarily be separated. This injects serious complications into the plan to produce a homeomorphism by a process which deals alternatively with point images and point inverses. Fortuitously, we find that we need not focus on approximating the original map by a homeomorphism. Instead, as is shown below, in the reduction of Theorem 2 to Theorem 2B, it suffices to approximate the inverse of the original map by a special kind of map, called an "acceptable" map. As a result, we can concentrate on eliminating point inverses, and we can ignore point images. Our inability to separate point images and point inverses will not hamper us, because we shall apply the replication device (repeatedly) on "one side" only. (Since we wish to excise point inverses, we apply the replication device on the left or domain-side of the relation.) (We shall find it necessary to preserve the disjointness of the point images and point inverses for technical reasons, to insure that the map which is the limit of infinitely many left-sided applications of the replication device is acceptable.) Thus, at the expense of adding another reduction step to the proof, we are able to get by with repeated applications of the replication device on one side only, and we avoid having to separate point images and point inverses. The observation that infinitely many left-sided applications of the replication device lead to a map approximating the inverse of the original map is due to R. D. Edwards. It is this observation which makes it possible to replace the hypothesis that the singular set of the original map is nowhere dense by the bald spot hypothesis.

In Theorem 2, we have replaced the hypothesis that $\{f^{-1}(y): y \in S(f)\}$ be a null collection by the weaker hypothesis that $S(f)$ be countable. This is an advantage, because the countability of $S(f)$ is the easier of the two hypothesis to detect and to preserve throughout the inductive process of the proof. Furthermore, the weaker hypothesis poses no additional difficulty in the proof for the following reason. Let $f: X \to Y$ be a map between compact spaces, let $\varepsilon > 0$, and consider the compact set $\{y \in S(f): \operatorname{diam} f^{-1}(y) \geq \varepsilon\}$. Under the stronger hypothesis, this set is finite; while under the weaker hypothesis, this set is compact and countable. We must deal with such a set in the proof of the Replication Lemma, where we must enclose it in the union of a finite number of small disjoint round n-cells. Fortunately, this can be accomplished for a compact countable set almost as easily as it can for a finite set.

The notions of "acceptable map" and "acceptable relation" appear in the proof of Theorem 2 in roles corresponding to those played by "admissible map" and "admissible relation" in the proof of Theorem 1. A map $f: B^n \to B^n$ is <u>acceptable</u> if $f|\partial B^n = 1|\partial B^n$ and $S(f)$ is a countable subset of $\operatorname{int} B^n$.

Theorem 2 reduces to:

THEOREM 2A. <u>Every acceptable map</u> $f: B^n \to B^n$ <u>can be approximated by homeomorphisms</u>.

PROOF THAT THEOREM 2A IMPLIES THEOREM 2. This proof is essentially the same as the proof that Theorem 1A implies Theorem 1. In this case, to locate a small collared n-cell C in the complement of the closure of the singular set, one uses the bald spot hypothesis rather than the nowhere density of the singular set. ⦀

Theorem 2A, in turn, reduces to:

THEOREM 2B. If $f: B^n \to B^n$ is an acceptable map and N is a neighborhood of f in $B^n \times B^n$, then there is an acceptable map $g: B^n \to B^n$ such that $g^{-1} \subset N$.

Theorem 2A is proved by repeated application of Theorem 2B, the output of Theorem 2B at one stage being used as the input at the next. Thus, the essential property of the map g produced by Theorem 2B is that it is acceptable. Indeed, general principles tell us that since the acceptable map $f: B^n \to B^n$ is cell-like, it is a fine homotopy equivalence [H] and automatically gives rise to a map $g: B^n \to B^n$ such that $g^{-1} \subset N$. However, this information is of no use in proving Theorem 2A unless g is known to be acceptable.

PROOF OF THEOREM 2A FROM THEOREM 2B. Assume Theorem 2B. Let $f: B^n \to B^n$ be an acceptable map. Let $\varepsilon > 0$. Set $f_0 = f$ and $N_0 = \{(x,y) \in B^n \times B^n: |f(x) - y| \leq \varepsilon\}$. N_0 is a closed neighborhood of f_0 in $B^n \times B^n$. We seek a homeomorphism $h: B^n \to B^n$ such that $h \subset N_0$. To this end, we shall construct a sequence $\{f_i\}$ of acceptable maps from B^n to itself, and a sequence $\{N_i\}$ of closed subsets of $B^n \times B^n$ such that the following conditions hold.

(1_i) $f_i^{-1} \subset \operatorname{int} N_{i-1}$.

(2_i) N_i is a closed neighborhood of f_i in $B^n \times B^n$ such that $N_i \subset N_{i-1}^{-1}$ and $\operatorname{diam} N_i(x) < 1/i+1$ for every $x \in B^n$.

We already have f_0 and N_0. We proceed inductively. Let $i \geq 1$ and assume we have an acceptable map $f_{i-1}: B^n \to B^n$ and a closed neighborhood N_{i-1} of f_{i-1} in $B^n \times B^n$. We apply Theorem 2B to obtain an acceptable map $f_i: B^n \to B^n$ such that $f_i^{-1} \subset \operatorname{int} N_{i-1}$. Since $\operatorname{diam} f_i(x) = 0$ for every $x \in B^n$, then Lemma 2 provides a closed neighborhood N of f_i in $B^n \times B^n$ such that $\operatorname{diam} N(x) < 1/i+1$ for every $x \in B^n$. Set $N_i = N \cap (N_{i-1}^{-1})$. Then f_i and N_i satisfy (1_i) and (2_i).

Clearly $N_0 \supset N_2 \supset N_4 \supset \cdots$ is a decreasing sequence of closed subsets of $B^n \times B^n$. Also for every $i \geq 2$ and every $x \in B^n$, since $f_i(x)$ and $f_i^{-1}(x)$ are non-empty, then (2_i) implies that $N_i(x)$ and $N_i^{-1}(x)$ are non-empty subsets of diameter $< 1/i$. It follows that $h = \cap_{i=0}^{\infty} N_{2i}$ is a homeomorphism of B^n which lies in N_0. ⦀

As the discussion at the beginning of this section suggests, the central geometric idea of the proof of Theorem 2 is, as before, a replication device. This device is codified by the following lemma. Notice that the direction of the homeomorphism χ is the opposite of its direction in Lemma 3.

LEMMA 5 (THE REPLICATION DEVICE). *Suppose* $\varphi: B^n \to B^n$ *is an acceptable map*, C *and* D *are each the union of a finite number of disjoint round n-cells in* $\text{int}\, B^n$, *and* T *is a countable subset of* $\text{int}\, C$ *such that* $D \subset \text{int}\, C$ *and* $S(\varphi) \cap \partial D = \emptyset$. *Then there is an acceptable map* $\psi: B^n \to B^n$ *and a homeomorphism* $\chi: \psi^{-1}D \to \varphi^{-1}D$ *such that* $\varphi \circ \chi = \psi|\psi^{-1}D$, $\psi(\text{int}\, C) = \text{int}\, C$, ψ *restricts to the identity on* $B^n - \text{int}\, C$, *and* $\{S(\psi) \cup \psi(T)\} \cap [\partial D \cup (S(\varphi) - D)] = \emptyset$.

PROOF. $C = \cup_{i=1} C_i$ where each C_i is a round n-cell in $\text{int}\, B^n$. As in the proof of Lemma 3, for each i, $1 \leq i \leq k$, $\psi|C_i: C_i \to C_i$ will be a miniaturized replica of ψ.

Let $1 \leq i \leq k$. Let $D_i = D \cap C_i$ and $T_i = T \cap C_i$. We begin with a homeomorphsim $\sigma_i: C_i \to B^n$ such that $\sigma_i|D_i = 1|D_i$. Since $S(\varphi)$ and $\sigma_i^{-1}(S(\varphi))$ are countable, then we can apply Lemma 1 in $C_i - \text{int}\, D_i$ to obtain a homeomorphism λ_i of C_i which restricts to the identity on $D_i \cup (\partial C_i)$ such that $\lambda_i(\sigma_i^{-1}(S(\varphi))) \cap (S(\varphi) - D) = \emptyset$. Then $\lambda_i(\sigma_i^{-1}(S(\varphi))) \cap \partial D = \emptyset$, because $S(\varphi) \cap \partial D = \emptyset$ and σ_i and λ_i fix ∂D. We now define the homeomorphism $\tau_i: C_i \to B^n$ by $\tau_i = \sigma_i \circ \lambda_i^{-1}$. Then $\tau_i|D_i = 1|D_i$ and $\tau_i^{-1}(S(\varphi)) \cap [\partial D \cup (S(\varphi) - D] = \emptyset$. Since $S(\tau_t^{-1} \circ \varphi \circ \tau_i) = \tau_i^{-1}(S(\varphi))$, it follows that $(\tau_i^{-1} \circ \varphi \circ \tau_i)^{-1}[\partial D \cup (S(\varphi) - D)]$ is the union of a finite number of $(n-1)$-spheres and a countable set. Hence we can apply Lemma 1 in C_i to obtain a homeomorphism μ_i of C_i which restricts to the identity on ∂C_i such that $\mu_i(T_i)$ is disjoint from $(\tau_i^{-1} \circ \varphi \circ \tau_i)^{-1}[\partial D \cup (S(\varphi) - D]$. Consequently, $(\tau_i^{-1} \circ \varphi \circ \tau_i \circ \mu_i)(T_i) \cap [\partial D \cup (S(\varphi) - D)] = \emptyset$.

Define the map $\psi: B^n \to B^n$ by

$$\psi = \begin{cases} \tau_i^{-1} \circ \varphi \circ \tau_i \quad \mu_i & \text{on } C_i \text{ for } 1 \leq i \leq k \\ 1 \text{ on } B^n - \text{int}\, C \end{cases}$$

Since $S(\psi) = \cup_{i=1} \tau_i^{-1}(S(\varphi))$, it is easily verified that ψ is an acceptable map, and that $[S(\psi) \cup \psi(T)] \cap [\partial D \cup (S(\varphi) - D] = \emptyset$.

Since $\tau_i \circ \mu_i(\psi^{-1}D_i) = \varphi^{-1}D_i$ for $1 \leq i \leq k$, then a homeomorphism $\chi: \psi^{-1}D \to \varphi^{-1}D$ is defined by setting $\chi|\psi^{-1}D_i = \tau_i \circ \mu_i|\psi^{-1}D_i$ for $1 \leq i \leq k$. Clearly $\varphi \circ \chi = \psi|\psi^{-1}D$. ⁞⁞⁞

PROOF OF THEOREM 2B. The proof is inductive, and the induction step is isolated in Lemma 6 below.

We first describe the strategy of the proof. Let $f: B^n \to B^n$ be an acceptable map, and let N_0 be a closed neighborhood of f in $B^n \times B^n$. We seek an acceptable map $g: B^n \to B^n$ whose inverse lies in N_0. To obtain g, we first construct a decreasing sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of closed subsets of $B^n \times B^n$ such that for each $i \geq 1$:

(1) $N_i|\partial B^n = 1|\partial B^n$,

(2) $N_i^{-1}(y)$ is a non-empty set of diameter $< 1/i$ for every $y \in B^n$, and

(3) $\{x \in B^n: \operatorname{diam} N_i(x) \geq 1/i\}$ is a countable set.

Then we set $g = (\cup_{i=0} N_i)^{-1}$. Condition (2) forces g to be a function from B^n to itself. g is continuous because it is a closed subset of $B^n \times B^n$. Condition (1) implies that $g|\partial B^n = 1|\partial B^n$ and $S(g) \subset \operatorname{int} B^n$. Since $S(g) \subset \cup_{i=1}\{x \in B^n: \operatorname{diam} N_i(x) \geq 1/i\}$, then condition (3) forces $S(g)$ to be a countable set. We conclude that g is an acceptable map. Obviously $g^{-1} \subset N_0$.

Before proceeding with the details of the proof we establish the definition of "acceptable relation" and several other convenient bits of notation. Notice that in passing from admissible relations to acceptable relations, h changes from a homeomorphism to a map and its direction is reversed. A relation $R \subset B^n \times B^n$ is <u>acceptable</u> if

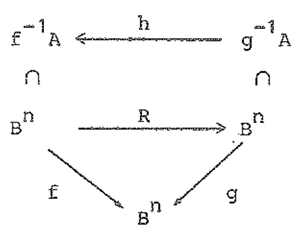
$$R = h^{-1} \cup g^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A)$$

where

(1) $f: B^n \to B^n$ and $g: B^n \to B^n$ are acceptable maps,

(2) A is the union of a finite number of disjoint round n-cells in $\operatorname{int} B^n$ such that $(S(f) \cup S(g)) \cap \partial A = \emptyset$ and $(S(f) - A) \cap (S(g) - A) = \emptyset$, and

(3) $h: g^{-1}A \to f^{-1}A$ is a map such that $f \circ h = g|g^{-1}A$ and $S(h)$ is a countable subset of $f^{-1}(\operatorname{int} A)$.



Let $R \subset X \times Y$ be a relation. Define

$$\sigma(R) = \cup\{R^{-1}(y): y \in Y \text{ and } R^{-1}(y) \text{ contains more than one point}\}$$

and define

$$\tau(R) = \{x \in X: R(x) \text{ contains more than one point}\}.$$

Now let $R \subset B^n \times B^n$ be an acceptable relation. Then $R = h^{-1} \cup g^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A)$ where f, g, h and A are as prescribed in the direction of "acceptable relation". We make four observations.

(1) R is a closed subset of $B^n \times B^n$

(2) $R|\partial B^n = 1|\partial B^n$

(3) $\sigma(R)$, $\tau(R)$ and ∂B^n are all disjoint.

(4) For each $\varepsilon > 0$, $\{x \in B^n: \operatorname{diam} R(x) \geq \varepsilon\}$ is a compact countable set.

The first observation is valid for the same reason that an admissible relation is a closed set. Observation (2) is clear. The third observation follows from the equations: $\sigma(R) = f^{-1}(S(f) - A)$ and $\tau(R) = S(h) \cup f^{-1}(S(g) - A)$. It follows that $\sigma(R) \cup \tau(R) \subset \operatorname{int} B^n$. Also since $(S(f) - A) \cap (S(g) - A) = \emptyset$ and $S(h) \subset f^{-1}(A)$, it is clear that $\sigma(R) \cap \tau(R) = \emptyset$. To prove observation (4), note that $\{x \in B^n: \operatorname{diam} R(x) \geq \varepsilon\}$ is the union of the two sets $\{x \in S(h): \operatorname{diam} h(x) \geq \varepsilon\}$ and $f^{-1}(\{z \in S(g): \operatorname{diam} g^{-1}(z) \geq \varepsilon\} - \operatorname{int} A)$. These two sets are compact and countable because $S(h)$ and $S(g)$ are countable and $(S(f) - A) \cap (S(g) - A) = \emptyset$.

We now give the details of the proof. Set $R_0 = f$; then R_0 is an acceptable relation (with $g = 1|B^n$, $A = \emptyset$ and $h = \emptyset$). The closed neighborhood N_0 of R_0 is given. We shall construct a sequence $\{R_i\}$ of acceptable relations in $B^n \times B^n$ and a sequence $\{N_i\}$ of closed subsets of $B^n \times B^n$ such that for each $i \geq 1$, the following conditions hold.

(1_i) $R_i \subset N_{i-1}$, $R_i|\sigma(R_i) \subset \operatorname{int} N_{i-1}$ and $\operatorname{diam} R_i^{-1}(y) < 1/i$ for every $y \in B^n$.

(2_i) $R_i \subset N_i \subset N_{i-1}$, $R_i|\sigma(R_i) \subset \operatorname{int} N_i$, $N_i|\partial B^n = 1|\partial B^n$, $\operatorname{diam} N_i^{-1}(y) < 1/i$ for every $y \in B^n$, and $\{x \in B^n: \operatorname{diam} N_i(x) \geq 1/i\}$ is a countable set.

R_0 and N_0 are given. We proceed inductively. Let $i \geq 1$ and assume we have an acceptable relation R_{i-1} and a closed subset N_{i-1} of $B^n \times B^n$ such that $R_{i-1} \subset N_{i-1}$ and $R_{i-1}|\sigma(R_{i-1}) \subset \operatorname{int} N_{i-1}$. We apply Lemma 6 below to obtain R_i satisfying (1_i), by substituting $(R_{i-1}, 1/i, N_{i-1})$ for (R, ε, N). Then Lemma 6 produces R_*, and we set $R_i = R_*$.

To obtain N_i satisfying (2_i), we must apply Lemma 2 twice. First, since $\operatorname{diam} R_i^{-1}(y) < 1/i$ for every $y \in B^n$, Lemma 2 provides a closed neighborhood L of R_i^{-1} in $B^n \times B^n$ such that $\operatorname{diam} L(y) < 1/i$ for every $y \in B^n$. For the second application of Lemma 2, we set

$$T = \{x \in B^n: \operatorname{diam} R_i(x) \geq 1/i\} \cup \partial B^n .$$

Since $\{x \in B^n: \operatorname{diam} R_i(x) \geq 1/i\}$ is compact, then T is a closed subset of B^n. Also $\sigma(R_i) \subset B^n - T$ because $T \subset \tau(R_i) \cup \partial B^n$. Lemma 2 now provides a closed subset M of $B^n \times B^n$ such that $R_i|B^n - T \subset \operatorname{int} M$, $\operatorname{diam} M(x) < 1/i$ for every $x \in B^n - T$, and $M|T = R_i|T$. It follows that $R_i|\sigma(R_i) \subset \operatorname{int} M$ because $\sigma(R_i) \subset B^n - T$, and that $M|\partial B^n = 1|\partial B^n$ because $\partial B^n \subset T$ and $R_i|\partial B^n = 1|\partial B^n$. Thus $\{x \in B^n: \operatorname{diam} M(x) \geq 1/i\}$ coincides with the countable set $\{x \in B^n: \operatorname{diam} R_i(x) \geq 1/i\}$. We conclude that ($2_i$) is satisfied if we set $N_i = L^{-1} \cap M \cap N_{i-1}$.

Let $i \geq 1$. Note that $R_i^{-1}(y) \neq \emptyset$ for every $y \in B^n$ because R_i is acceptable. Thus, (2_i) implies that $N_i^{-1}(y)$ is non-empty and of diameter $< 1/i$

for every $y \in B^n$. Also $N_i \subset N_{i-1}$, $N_i|\partial B^n = 1|\partial B^n$, and $\{x \in B^n : \operatorname{diam} N_i(x) \geq 1/i\}$ is a countable set. Now, as we argued earlier, an acceptable map $g : B^n \to B^n$ such that $g^{-1} \subset N_0$ is specified by $g = (\bigcap_{i=0}^{\infty} N_i)^{-1}$. ∎

LEMMA 6. <u>If</u> $R \subset B^n \times B^n$ <u>is an acceptable relation,</u> $\varepsilon > 0$ <u>and</u> N <u>is a closed subset of</u> $B^n \times B^n$ <u>such that</u> $R \subset N$ <u>and</u> $R|\sigma(R) \subset \operatorname{int} N$, <u>then there is an acceptable relation</u> $R_* \subset B^n \times B^n$ <u>such that</u> $\operatorname{diam} R_*^{-1}(y) < \varepsilon$ <u>for every</u> $y \in B^n$, $R_* \subset N$ <u>and</u> $R_*|\sigma(R_*) \subset \operatorname{int} N$.

PROOF. Since R is acceptable then $R = h^{-1} \cup g^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A)$ where f, g, h and A are as prescribed in the definition of "acceptable relation". Let $Z = \{z \in S(f) : \operatorname{diam} f^{-1}(z) \geq \varepsilon\} - A$. Z is a compact countable subset of $\operatorname{int} B^n - A$ because $S(f)$ is a countable subset of $\operatorname{int} B^n - \partial A$. The significance of Z is that $\{f^{-1}(z) : z \in Z\} = \{R^{-1}(y) : y \in B^n$ and $\operatorname{diam} R^{-1}(y) \geq \varepsilon\}$, and the latter set is precisely the set of point inverses of R whose diameter must be reduced.

We proceed as we did in the proof of Lemma 4. We enclose Z in the union D of a finite number of small disjoint round n-cells in $\operatorname{int} B^n$. Then we use the Replication Device (Lemma 5) to modify g so that there is a natural map from $g^{-1}D$ to $f^{-1}D$. We can then alter R on $f^{-1}D$ so that $R|f^{-1}D$ is the inverse of this map, thereby eliminating all the non-trivial point inverses of R arising from points of $g^{-1}D$. In particular, this eliminates all point inverses of R of diameter $\geq \varepsilon$.

For each $z \in Z$, since $f^{-1}(z) \times g^{-1}(z) = R|f^{-1}(z) \subset R|\sigma(R) \subset \operatorname{int} N$, then z has a neighborhood U_z in $\operatorname{int} B^n - A$ such that $f^{-1}U_z \times g^{-1}U_z \subset \operatorname{int} N$. We now begin choosing a sequence $C_1, C_2, C_3, \ldots$ of disjoint round n-cells in $\operatorname{int} B^n$ such that for each $i \geq 1$, $\partial C_i \cap Z = \emptyset$ and $z \in \operatorname{int} C_i \subset C_i \subset U_z$ for some $z \in Z$. Since Z is countable, we can continue to choose C_i's for as long as some points of Z remain uncovered. However, since Z is compact, this process must terminate after a finite number of choices, yielding a finite collection $C_1, C_2, \ldots, C_k$ of disjoint round n-cells in $\operatorname{int} B^n$ such that if $C = \bigcup_{i=1} C_i$, then $Z \subset \operatorname{int} C$, $C \cap A = \emptyset$ and $f^{-1}C_i \times g^{-1}C_i \subset \operatorname{int} N$ for $1 \leq i \leq k$. (The third condition will be used to insure that $R_* \subset N$ and $R_*|\sigma(R_*) \subset \operatorname{int} N$.) Since $S(f)$ is a countable set, then for each i, $1 \leq i \leq k$, there is a round n-cell D_i such that $D_i \subset \operatorname{int} C_i$, and if $D = \bigcup_{i=1} D_i$, then $Z \subset \operatorname{int} D$ and $S(f) \cap \partial D = \emptyset$.

We now apply Lemma 5 with f in the role of φ and $S(g) \cap \operatorname{int} C$ in the role of T. We obtain an acceptable map $\psi : B^n \to B^n$ and a homeomorphism $\chi : \psi^{-1}D \to f^{-1}D$ such that $f \circ \chi = \psi|\psi^{-1}D$, $\psi(\operatorname{int} C) = \operatorname{int} C$, $\psi = 1$ on $B^n - \operatorname{int} C$, and $[S(\psi) \cup \psi(S(g) \cap \operatorname{int} C)] \cup [\partial D \cup (S(f) - D)] = \emptyset$. At this point, it is convenient to observe that since $\psi(S(g) - \operatorname{int} C) = S(g) - \operatorname{int} C$, and the latter set is disjoint from both ∂A and $S(f) - A$, then $S(\psi) \cup \psi(S(g))$ is disjoint from both $\partial(A \cup D)$ and $S(f) - (A \cup D)$. Also note that $S(f) \cap \partial(A \cup D) = \emptyset$.

We define the map $g_*: B^n \to B^n$ by $g_* = \psi \circ g$. Since $S(g_*) = S(\psi) \cup \psi(S(g))$, then g_* is evidently an acceptable map.

We set $A_* = A \cup D$. Then A_* is the union of a finite number of disjoint round n-cells in $\operatorname{int} B^n$. It follows from our observations above that $(S(f) \cup S(g_*)) \cap \partial A_* = \emptyset$ and $(S(f) - A_*) \cap (S(g_*) - A_*) = \emptyset$.

Since $g_*^{-1}A_* = g^{-1}A \cup g^{-1}(\psi^{-1}D)$, then a map $h_*: g_*^{-1}A_* \to f^{-1}A_*$ is defined by setting $h_*|g^{-1}A = h$ and $h_*|g^{-1}(\psi^{-1}D) = \chi \circ g|g^{-1}(\psi^{-1}D)$. It is easy to check that $f \circ h_* = g_*|g_*^{-1}A_*$. Since $S(h_*) = S(h) \cup \chi(S(g|g^{-1}(\psi^{-1}D))$ and $\psi(S(g)) \cap \partial D = \emptyset$, then $S(h_*)$ is a countable subset of $f^{-1}(\operatorname{int} A_*)$.

Now we can define an acceptable relation $R_* \subset B^n \times B^n$ by the formula $R_* = h_*^{-1} \cup g_*^{-1} \circ f|f^{-1}(B^n - \operatorname{int} A_*)$.

It follows that $R_*^{-1} = h_* \cup f^{-1} \circ g_*|g_*^{-1}(B^n - \operatorname{int} A_*)$. Now suppose $y \in B^n$ and $\operatorname{diam} R_*^{-1}(y) > 0$. Then $y \in g_*^{-1}(B^n - \operatorname{int} A_*)$ and $R_*^{-1}(y) = f^{-1}(g_*(y))$. Z, D and A_* are chosen so that $\{z \in S(f): \operatorname{diam} f^{-1}(z) \geq \varepsilon\} \subset \operatorname{int} A_*$. Since $g_*(y) \notin \operatorname{int} A_*$, it follows that $\operatorname{diam} f^{-1}(g_*(y)) < \varepsilon$. Thus $\operatorname{diam} R_*^{-1}(y) < \varepsilon$.

Lastly, we demonstrate that $R_* \subset N$ and $R_*|\sigma(R_*) \subset \operatorname{int} N$. Since $g_*^{-1} = g^{-1}$ on $B^n - \operatorname{int} C$ and $h_*^{-1} = h^{-1}$ on $f^{-1}A$, it follows that $R_*|f^{-1}(B^n - \operatorname{int} C) = R|f^{-1}(B^n - \operatorname{int} C) \subset N$. Also the equation $f \circ h_* = g_*|g_*^{-1}A_*$ implies that $h_*^{-1} \subset g_*^{-1} \circ f$, from which we deduce that $R_* \subset g_*^{-1} \circ f$. For $1 \leq i \leq k$, since $\psi(C_i) = C_i$, then $g_*^{-1}(C_i) = g^{-1}(C_i)$. Therefore, for $1 \leq i \leq k$,

$$R_*|f^{-1}C_i \subset g_*^{-1} \circ f|f^{-1}C_i \subset f^{-1}C_i \times g_*^{-1}C_i = f^{-1}C_i \times g^{-1}C_i \subset \operatorname{int} N .$$

Consequently, $R_*|f^{-1}C \subset \operatorname{int} N$. It is now evident that $R_* \subset N$. Since $\sigma(R) = f^{-1}(S(f) - A)$ and $\sigma(R_*) = f^{-1}(S(f) - A_*)$, then $\sigma(R_*) \subset \sigma(R)$. Thus,

$$R_*|\sigma(R_*) - f^{-1}C = R|\sigma(R_*) - f^{-1}C \subset R|\sigma(R) \subset \operatorname{int} N.$$

Since $R_*|\sigma(R_*) \cap f^{-1}(C) \subset R_*|f^{-1}(C) \subset \operatorname{int} N$, we conclude that $R_*|\sigma(R_*) \subset \operatorname{int} N$. |||

5. TAME ZERO-DIMENSIONAL SINGULAR SETS

We shall deduce Theorem 3 from Theorem 2 by passing from a map with a tame zero-dimensional singular set to a map with a countable singular set. This transformation requires two propositions. The first is that any σ-compact tame zero-dimensional set can be enclosed in a null collection of small disjoint collared n-cells. This fact is established below in Lemma 7. The second is a fundamental decomposition shrinking principle which originates in the work of R. H. Bing, and is known as "the Null Star-like Equivalent Shrinking Principle". It applies here to show that a decomposition of an n-manifold determined by a null collection of disjoint collared n-cells is shrinkable. We describe this principle in more detail below.

Lemma 7 captures the fundamental properties of tame zero-dimensional sets. Before presenting this lemma, we feel it appropriate to comment on the definition of "tame zero-dimensionality". Let M be a compact n-manifold. One of the classical definitions of zero-dimensionality implies that a subset S of M is zero-dimensional if every point of S has arbitrarily small neighborhoods in M whose frontiers miss S. The definition of tame zero-dimensionality applies only to σ-compact subsets of $\operatorname{int} M$; recall that it states that a σ-compact subset S of $\operatorname{int} M$ is tame zero-dimensional if each point of S has arbitrarily small collared n-cell neighborhoods in M whose boundaries miss S. Clearly, the definition of tame zero-dimensionality makes sense for arbitrary (not just σ-compact) subsets of $\operatorname{int} M$, and comparison with the above classical definition of zero-dimensionality tempts us to drop the restriction to σ-compacta. We resist this temptation for the following reason. Originally a subset of manifold was called "tame" if it behaved like a piecewise linearly embedded polyhedron of the same dimension. Thus, a tame zero-dimensional subset should behave in some sense like a finite set of points. As the level of understanding of tame sets rose, it was recognized that the specific properties which tame sets share with piecewise linearly embedded polyhedra of the same dimension are their general position properties. For a tame zero-dimensional set, the appropriate general position property is expressed below in statement (2) of Lemma 7. This general position property can be proved for tame zero-dimensional σ-compacta. However, it is not necessarily valid for arbitrary subsets of $\operatorname{int} M$ which satisfy the definition of tame zero-dimensionality. An illustration of this phenomenon is given in the next paragraph. For this reason, we do not use the term "tame zero-dimensional" outside the class of σ-compacta.

Let $J = \{(x,y,z) \in \mathbb{R}^3 : x, y \text{ and } z \text{ are irrational}\}$. J is not σ-compact. However J satisfies the definition of tame zero-dimensionality, because any prism of the form $[a,b] \times [c,d] \times [e,f]$ where a,b,c,d,e and f are rational, is a collared 3-cell whose boundary misses J. Let A be the Cantor set in $\mathbb{R}^3$ known as Antoine's necklace. A is a compact wild (= not tame) zero-dimensional nowhere dense subset of $\mathbb{R}^3$ with the following property. Every non-empty open subset of A contains a wild Cantor set - in fact, a smaller copy of A. We assert that no homeomorphism of $\mathbb{R}^3$ carries J off A. Thus J does not possess the general position property which characterizes tame zero-dimensional σ-compacta. For a simple proof by contradiction, suppose h is a homeomorphism of $\mathbb{R}^3$ such that $h(J) \cap A = \emptyset$. Then $h^{-1}A \subset \mathbb{R}^3 - J$. Since $\mathbb{R}^3 - J$ is the union of countably many flat 2-dimensional planes, the Baire Category Theorem implies that some non-empty open subset U of $h^{-1}A$ must lie in one of these planes. Since any Cantor set which lies in a flat 2-dimensional plane is tame in R^3, then U contains no wild Cantor sets. Hence, hU is a non-empty open subset of A which contains no wild Cantor sets.

LEMMA 7. Let S be a σ-compact subset of the interior of a compact manifold M. The following three statements are equivalent.

(1) S is tame zero-dimensional.

(2) If T is the union of a countable number of nowhere dense subsets of M, then $1|M$ can be approximated by homeomorphisms h of M such that $h(S) \cap T = \emptyset$ and $h|\partial M = 1|\partial M$.

(3) For every $\epsilon > 0$, there is a null collection $\{C_i\}$ of disjoint collared n-cells of diameter $< \epsilon$ in $\operatorname{int} M$ such that $S \subset \bigcup_{i=1} \operatorname{int} C_i$.

PROOF. (1) implies (2). Assume statement (1). We first establish statement (2) in the special case that S is compact and T is nowhere dense.

Let $\epsilon > 0$. Since S is compact, it is covered by a finite collection $\{K_i : 1 \leq i \leq p\}$ of collared n-cells of diameter $< \epsilon$ in $\operatorname{int} M$ such that $S \cap \partial K_i = \emptyset$ for $1 \leq i \leq p$. For $1 \leq i \leq p$, let $L_i = K_i - \bigcup_{j<i} \operatorname{int} K_j$. Then $\{\operatorname{int} L_i : 1 \leq i \leq p\}$ is a cover of S by disjoint open sets of diameter $< \epsilon$.

Let $1 \leq i \leq p$. Set $S_i = S \cap L_i$. S_i is a compact subset of $\operatorname{int} L_i$. Hence, S_i is covered by a finite collection $\{C_{i,j} : 1 \leq j \leq q(i)\}$ of collared n-cells in $\operatorname{int} L_i$ such that $S_i \cap \partial C_{i,j} = \emptyset$ for $1 \leq j \leq q(i)$, and $\{C_{i,j} : 1 \leq j \leq q(i)\}$ is irreducible in the sense that no proper subcollection covers S_i. For each j, $1 \leq j \leq q(i)$, there are collared n-cells $D_{i,j}$ and $E_{i,j}$ and a homeomorphism $h_{i,j}$ of M such that

(a) $E_{i,j} \subset \operatorname{int} D_{i,j} \subset D_{i,j} \subset \operatorname{int} C_{i,j}$,

(b) $S_i \cap (C_{i,j} - \operatorname{int} D_{i,j}) = \emptyset$,

(c) $E_{i,j}$ is disjoint from $C_{i,k}$ whenever $k \neq j$ for $1 \leq k \leq q(i)$, and $E_{i,j} \cap T = \emptyset$.

(d) $h_{i,j}(D_{i,j}) = E_{i,j}$ and $h_{i,j}|M - \operatorname{int} C_{i,j} = 1|M - \operatorname{int} C_{i,j}$.

Define the homeomorphism h_i of M by $h_i = h_{i,q(i)} \circ \cdots \circ h_{i,2} \circ h_{i,1}$. Then $h_i|M - \operatorname{int} L_i = 1|M - \operatorname{int} L_i$; so h_i is within ϵ of $1|M$. Also we assert that $h_i(S_i) \subset \bigcup_{j=1}^{q(i)} E_{i,j}$. To prove this, let $x \in S_i$. Choose j, $1 \leq j \leq q(i)$, so that $x \in C_{i,j}$ and $x \notin C_{i,k}$ for $1 \leq k < j$. Then $h_{i,k}$ fixes x for $1 \leq k < j$. Also $x \in D_{i,j}$, so that $h_{i,j}(x) \in E_{i,j}$. Consequently, $h_{i,k}$ fixes $h_{i,j}(x)$ for $j < k \leq q(i)$. It follows that $h_i(x) = h_{i,j}(x) \in E_{i,j}$. Since each $E_{i,j}$ misses T, we have that $h_i(S_i) \cap T = \emptyset$.

Now we define the homeomorphism h of M by setting $h|L_i = h_i|L_i$ for $i \leq i \leq p$ and setting $h|M - \bigcup_{i=1} \operatorname{int} L_i = 1|M - \bigcup_{i=1} \operatorname{int} L_i$. Then h is within ϵ of $1|M$, $h(S) \cap T = \emptyset$ and $h|\partial M = 1|\partial M$. This finishes the proof of statement (2) in the special case.

To prove statement (2) in the general case, we write $S = \bigcup_{i=1}^{\infty} S_i$ and $T = \bigcup_{j=1}^{\infty} T_j$ where each S_i is compact and each T_j is nowhere dense.

For each $i \geq 1$ and $j \geq 1$, let $U_{i,j} = \{h \in \mathcal{H}(M,\partial M): h(S_i) \cap c\ell T_j = \emptyset\}$. Since S is tame zero-dimensional, so is each S_i; hence $h(S_i)$ is tame zero-dimensional for each $i \geq 1$ and every $h \in \mathcal{H}(M,\partial M)$. Since each T_j is nowhere dense, so is each $c\ell T_j$. Therefore, we can deduce from the special case of statement (2) proved above, that each $U_{i,j}$ is a dense subset of $\mathcal{H}(M,\partial M)$. Also each $U_{i,j}$ is evidently an open subset of $\mathcal{H}(M,\partial M)$. Since $\mathcal{H}(M,\partial M)$ has a complete metric, we conclude via the Baire Category Theorem that $\cap_{i=1}^{\infty} {}_{j=1}^{\infty} U_{i,j}$ is a dense subset of $H(M,\partial M)$. Statement (2) now follows because $1|M$ can be approximated by elements of $\cap_{i=1}^{\infty} {}_{j=1}^{\infty} U_{i,j}$.

(2) implies (3). Assume statement (2). One can easily choose a null collection $\{C_i\}$ of disjoint collared n-cells of diameter $< \varepsilon/3$ in $\operatorname{int} M$ such that $\cup_{i=1} \operatorname{int} C_i$ is a dense subset of M. Then $M - \cup_{i=1} \operatorname{int} C_i$ is nowhere dense in M. Statement (2) provides a homeomorphism h of M within $\varepsilon/3$ of $1|M$ such that $h(S) \cap (M - \cup_{i=1} \operatorname{int} C_i) = \emptyset$ and $h|\partial M = 1|\partial M$. It follows that $\{h^{-1}C_i\}$ is a null collection of disjoint collared n-cells of diameter $< \varepsilon$ in $\operatorname{int} M$ whose interiors cover S.

(3) implies (1). Assume statement (3). Let $x \in S$ and let U be an open neighborhood of x in M. Choose $\varepsilon > 0$ so that ε is less than the distance from x to $M-U$. Statement (3) provides a null collection $\{C_i\}$ of disjoint collared n-cells of diameter $< \varepsilon$ in $\operatorname{int} M$ whose interiors cover S. Hence, $x \in \operatorname{int} C_i$ for some $i \geq 1$. Also $\partial C_i \cap S = \emptyset$. Since $\operatorname{diam} C_i < \varepsilon$, then $C_i \subset U$. This proves S is tame zero-dimensional. ▥

Perhaps the fundamental geometric tool of decomposition space theory is the Null Star-like Equivalent Shrinking Principle. A compact subset F of $\mathbb{R}^n$ is <u>star-like</u> if there is a point p in F such that every ray in $\mathbb{R}^n$ emanating from p intersects F in a connected set. A compact subset F of the interior of an n-manifold M is <u>star-like equivalent</u> if there is a neighborhood U of F in M and an embedding $e: U \to \mathbb{R}^n$ such that $e(F)$ is star-like. Observe that any collared n-cell in an n-manifold is star-like equivalent.

THE NULL STAR-LIKE EQUIVALENT SHRINKING PRINCIPLE. <u>Suppose</u> $f: M \to X$ <u>is a surjective map from a compact boundaryless manifold</u> M <u>to a compact metric space</u> X. <u>If</u> $\{f^{-1}(y): y \in S(f)\}$ <u>is a null collection of star-like equivalent sets, then</u> f <u>can be approximated by homeomorphisms</u>.

This principle has manifested itself in many forms, apparently originating in [B1], and playing major roles in a number of significant results including [C], [E] and [F].

PROOF OF THEOREM 3. Let $f: S^n \to S^n$ be a map with a bald spot and a tame zero-dimensional singular set. Let $\varepsilon > 0$. Then there is a collared n-cell D in S^n disjoint from $S(f)$, and Lemma 7 provides a null collection $\{C_i\}$ of disjoint collared n-cells of diameter $< \varepsilon$ in $S^n - D$ such that $S(f) \subset \cup_{i=1} \operatorname{int} C_i$.

Let $X = \{C_i : i \geq 1\} \cup \{\{y\} : y \in S^n - \cup_{i=1} C_i\}$; i.e., X is the quotient space obtained from S^n by identifying each C_i to a point. Let $\pi: S^n \to X$ denote the quotient map; thus $y \in \pi(y)$ for every $y \in S^n$. We endow X with the quotient topology. This makes $\pi: S^n \to X$ continuous and makes X a compact metric space. Notice that since $\{\pi^{-1}(x) : x \in S(\pi)\} = \{C_i : i \geq 1\}$, then the Null Star-like Equivalent Shrinking Principle asserts that $\pi: S^n \to X$ can be approximated by homeomorphisms. Consequently, X is homeomorphic to S^n.

Consider the map $\pi \circ f: S^n \to X$. Its singular set is the countable set $\{\pi(C_i) : i \geq 1\}$. Also it has a bald spot because $f|f^{-1}(\mathrm{int}\, D)$ and $\pi|\mathrm{int}\, D$ are homeomorphisms. Since X is homeomorphic to S^n, Theorem 2 implies that $\pi \circ f: S^n \to X$ can be approximated by homeomorphisms. (This procedure, which encloses $S(f)$ in the null collection $\{C_i\}$ to yield a map $\pi \circ f$ with a countable singular set, is called "amalgamation".)

Let d denote the given metric on S^n, and let d' be a metric on X. Since $\mathrm{diam}\, C_i < \varepsilon$ for each $i \geq 1$, then there is a $\delta > 0$ such that for all $y, z \in S^n$, if $d'(\pi(y), \pi(z)) < \delta$, then $d(y,z) < \varepsilon$. Let $g: S^n \to X$ and $h: S^n \to X$ be homeomorphisms such that g is within $\delta/2$ of π, and h is within $\delta/2$ of $\pi \circ f$. We assert that the homeomorphism $g^{-1} \circ h: S^n \to S^n$ is within ε of f. To see this, let $y \in S^n$. Then $d'(\pi \circ f(y), h(y)) < \delta/2$ and $d'(\pi(g^{-1} \circ h(y)), g(g^{-1} \circ h(y))) < \delta/2$. Hence $d'(\pi(f(y)), \pi(g^{-1} \circ h(y))) < \delta$. Therefore, the choice of δ insures that $d(f(y), g^{-1} \circ h(y)) < \varepsilon$. ∎

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